

Philosophy 2310
Evaluating Quantifier Sentences

One important aspect of understanding the language of FOL is to learn what kinds of situations make various sentences of FOL true or false. When we are dealing with situations that involve only a finite number of objects there is a straightforward computational procedure that can be used. I will explain this procedure through the use of a series of examples. I will use the following diagrams for the initial examples:

Diagram 1

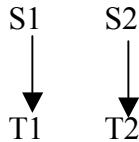


Diagram 2

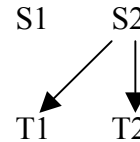


Diagram 3

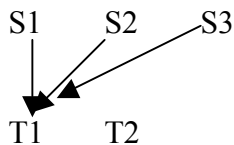
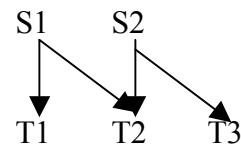


Diagram 4



EXAMPLE 1. $\forall x(S(x) \rightarrow \exists y(T(y) \wedge \neg P(x,y)))$: for every x , if x is a student, then there is a y such that y is a test and x did not pass y .

The main connective is the universal quantifier; so for the sentence to be true, it must be true in every case. Since anything which isn't an S will automatically make the conditional true, we can merely check all of the relevant cases – those that satisfy the antecedent. Since the antecedent of the conditional involves the “ S ” predicate (“student”), we look at the top of the diagram. In Diagram 1, there are two students $S1$ and $S2$; so for the sentence to be true, the consequent of the conditional must be true for both $S1$ and $S2$. This amounts to evaluating the consequent of the conditional with each of $S1$ and $S2$ in the place of “ x ”:

Case 1. $\exists y(T(y) \wedge \neg P(S1, y))$

Case 2. $\exists y(T(y) \wedge \neg P(S2, y))$

Each of these must be true for the original sentence to be true. Consider Case 1: here the question is whether there is a test that $S1$ did not pass. Since there is such a test (namely, $T2$), Case 1 is true. Consider Case 2: here the question is whether there is a test that $S2$ passed. Since there is such a test (namely $T1$), Case 2 is true. Since these are the only relevant cases and they are all true, Example 1 is true on Diagram 1. Note that the test chosen does not have to be the same in the two cases; that is the effect of having the existential quantifier occurring inside the scope of the universal quantifier.

Now let's look at the same sentence on Diagram 2. In order for it to be true, there are two relevant cases to check:

Case 1. $\exists y(T(y) \wedge \neg P(S1, y))$

Case 2. $\exists y(T(y) \wedge \neg P(S2, y))$

In Case 1, there is at least one test that S1 did not pass (both T1 and T2 qualify; only one is needed though). So Case 1 is true. On the other hand, since S2 passed both tests, Case 2 is false. Since the original claim is universal, it must be true in both cases; thus Example 2 is false in diagram 2.

In Diagram 3 there are three cases to consider. In addition to the same two cases as above, we also have to consider:

Case 3: $\exists y(Ty \wedge \neg P(S3, y))$

Case 1 is true since S1 does not pass T2. Case 2 is true since S2 does not pass T2 and Case 3 is true since S3 does not pass T2. Thus Example 1 is true on Diagram 3.

Example 1 is also true on Diagram 4, but I will not go through the reasoning.

Note that it is not necessary to translate the original sentence into English and then determine whether the English sentence is true on each of the diagrams. If the meaning of the sentence is absolutely clear to you it does not hurt; however, as quantifier sentences become more complex, translation back into English tends to become less helpful and is often confusing. Even if you can easily translate more complicated sentences, it is essential to learn the mechanics of determining the truth-value of the sentence on a diagram. Once you break a sentence down into pieces it becomes easier and thus more valuable to translate back. For example, in each Case above we have only one quantifier and I translated it into "there is at least one test that S1 does not pass" etc. At this lowest level translation is easy and often helpful.

EXAMPLE 2: $\exists x(T(x) \wedge \forall y(S(y) \rightarrow \neg P(y,x)))$: there is an x such that x is a test and for every y, if y is a student, then y did not pass x.

The sentence has an existential quantifier as its main connective; so in order for it to be true, it only needs to be true in one case (it's OK if there are more). Thus the existential quantifier works just like disjunction. Since the predicate immediately to the right of the main connective is "T" (for "test"), in the first three diagrams there are two cases to consider, T1 and T2. In the fourth diagram, there will be three. What follows the conjunction sign must be true in at least one of these cases for the sentence to be true. It is false only if it is false in every case.

Diagram 1: Case 1: $\forall y(S(y) \rightarrow \neg P(y, T1))$

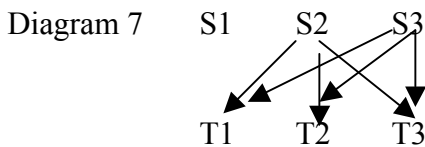
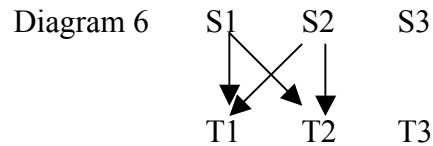
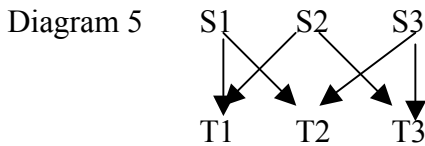
Case 2: $\forall y(S(y) \rightarrow \neg P(y, T2))$

Case 1 claims that for every student y , y did not pass T1; that is, no student passed T1. Thus is false because S1 passed T1. So Case 1 is false. Also, Case 2 is false; it claims that no student passed T2. But S2 passed T2. So both Case 1 and Case 2 are false. Therefore Example 2 is false on Diagram 1.

Example 2 is also false on Diagrams 2 and 4 since each of the cases are false.

However, Example 2 is true on Diagram 3. Here, Case 2 is true since no student passed T2. Since Case 2 is true and the original sentence was an existential sentence, the sentence is true.

More difficult examples:



EXAMPLE 3 $\forall x \forall y ((T(x) \wedge T(y) \wedge x \neq y) \rightarrow \exists z (S(z) \wedge P(z,x) \wedge P(z,y)))$; for every x and for every y , if x is a test and y is a test and x is not identical to y , then there is a z such that z is a student and z passed x and z passed y .

Here the initial quantifiers are universal; so the sentence must be true in all relevant cases. Here the cases to be considered are described by the antecedent " $T(x) \wedge T(y) \wedge x \neq y$ ". We must examine all cases where x is a test, y is a test and x and y are not identical. That means that we must examine all pairs of different tests. For each diagram there are six pairs to consider:

$\langle T1, T2 \rangle$ $\langle T1, T3 \rangle$ $\langle T2, T3 \rangle$ $\langle T2, T1 \rangle$ $\langle T3, T1 \rangle$ $\langle T3, T2 \rangle$

Notice that $\langle T1, T2 \rangle$ and $\langle T2, T1 \rangle$ are different pairs. In one case, T1 is being plugged in for x and T2 is being plugged in for y . In the other case, T2 is x and T1 is y . These are different cases.

Considering now the consequent of the conditional, this gives rise to six cases, all of which must come out true if the original sentence is to be true. The cases are:

- Case 1. $\exists y (S(y) \wedge P(y, T1) \wedge P(y, T2))$
- Case 2. $\exists y (S(y) \wedge P(y, T1) \wedge P(y, T3))$

- Case 3. $\exists y(S(y) \wedge P(y, T2) \wedge P(y, T1))$
Case 4. $\exists y(S(y) \wedge P(y, T2) \wedge P(y, T2))$
Case 5. $\exists y(S(y) \wedge P(y, T3) \wedge P(y, T1))$
Case 6. $\exists y(S(y) \wedge P(y, T3) \wedge P(y, T2))$

The first of these says that there is a student who passed both T1 and T2. This is true in Diagram 5 because of S1. The second case says that there is a student that passed both T1 and T3. This is true because S2 passed both T1 and T3. The third says that there is a student who passed both T2 and T3. This is true because S3 passed both T2 and T3. Cases 4, 5, and 6 are also true, and because of the symmetry of conjunction they depend on the same facts as the earlier cases. Thus in a sense, there are really only three pairs of tests to consider, but this is because of the particular example. To pass both T1 and T2 just is to pass both T2 and T1. However, in general the two tests might play different roles in a sentence so you need to also check each of these pairs as if they are different.

In Diagram 6, we need to consider the same six cases. Case 1 is true since at least one student passed both T1 and T2 (both S1 and S2 did). However, Case 2 is false, since no student passed T1 and T3. Since there is a false case and the original sentence is universal, Example 3 is false on Diagram 6.

Example 3 is true on Diagram 7 since for any pair of tests we can find a student who passed them both. In fact, if there is a student who passed every test (here S2 or S3) in each of the six cases, that same student will have passed both tests in question and so we know that each of the cases is true.

EXAMPLE 4. $\forall x((S(x) \wedge \exists y(T(y) \wedge P(x,y))) \rightarrow \forall y(T(y) \rightarrow P(x,y)))$; for every x, if x is a student and there is a y such that y is a test and x passed y, then for every y, if y is a test, then x passed y.

Here the sentence is universal, so the claim must be true in all relevant cases. What are the relevant cases? The antecedent of the conditional is “ $S(x) \wedge \exists y(T(y) \wedge P(x,y))$ ”; that is to say, all students who satisfy the added condition of being such that there is a test that they passed. In other words, we are checking to see if every student who passed at least one test passed every test.

Consider Diagram 7. In Diagram 7, there are three students, but only two of them, S2 and S3, meet the additional condition. So in evaluating Example 4 on Diagram 7, we only need to consider two cases:

- Case 1. $\forall y(T(y) \rightarrow P(S2, y))$
Case 2. $\forall y(T(y) \rightarrow P(S3, y))$

Both the cases are true since both S2 and S3 passed every test. Therefore the original sentence is true. However, Example 4 is false on both Diagram 5 and Diagram 6.